JOURNAL OF SPACECRAFT AND ROCKETS
Vol. 33, No. 3, May–June 1996

Dynamic Continuum Modeling of Truss-Type Space Structures Using Spectral Elements

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A new continuum modeling approach is developed by utilizing the spectral element method to represent a large periodic truss-type beam as an equivalent continuum extended Timoshenko beam model. First, the transfer matrix for a representative truss cell of the periodic truss-type beam is numerically derived by condensing the global dynamic stiffness matrix by use of the continuum degrees of freedom introduced. The global dynamic stiffness matrix is obtained by assembling the spectrally formulated dynamic stiffness matrices for the structural elements within the truss cell. Next, the transfer matrix for an equivalent continuum beam element is analytically derived so that the unknown equivalent structural properties are contained in the transfer matrix. The two transfer matrices are then forced to be equal to each other to determine the equivalent structural properties of the continuum beam model. The equivalent structural properties and vibration characteristics of a continuum beam model by the present continuum method are compared with those by other continuum methods to prove that the present continuum method can give a very reliable continuum model.

Nomenclature

A	= cross-sectional area of beam
$A(\omega)$	= matrix defined in Eq. (22)
a_{ij}, b_{ij}	= matrix components defined in Eq. (28)
C_{ij}	= coupling structural rigidities
$\boldsymbol{D}$	= nodal displacement vector for the continuum model
$\boldsymbol{d}$	= nodal displacement vector for the truss cell
E	= Young's modulus of elasticity
$\boldsymbol{F}$	= nodal force vector for the continuum model
$\boldsymbol{f}$	= nodal force vector for the truss cell
G	= shear modulus of elasticity
H	= thickness of truss cell; see Fig. 4
I	= second moment of area of beam cross section
$\boldsymbol{k}$	= dynamic stiffness matrix of beam element
k_b	= wave number of beam
k_r	= wave number of rod
L	= total length of beam
l	= length of beam element
M, F_z	= bending moments
N, F_x	= axial forces
Q, F_y	= transverse shear forces
$\boldsymbol{T}_d, \boldsymbol{T}_j$	= matrices defined in Eq. (16)
u, U	= axial displacements
v, V	= transverse displacements
$\boldsymbol{X}, \boldsymbol{Y}$	= quantities defined in Eq. (12)
$\boldsymbol{y}$	= state vector; see Eq. (23)
$Z(\omega)$	= global dynamic stiffness matrix
α, β, γ	= quantities defined in Eq. (8)
θ, Θ	= rotational displacements
$\rho A, \rho I, \rho R$	= structural inertia properties defined in Eq. (21)
$\boldsymbol{\Phi}(\omega)$	= transition matrix; see Eq. (25)
ω	= circular frequency, rad/s

Subscripts

l, r	= quantities at the left and right of beam element
ETB	= extended Timoshenko beam
truss	= truss-type beam
1, 2, 3, 4	= quantities at node numbers of 1, 2, 3 and 4

Superscript

$-$	= spectral components

Received April 21, 1995; revision received Dec. 7, 1995; accepted for publication Dec. 12, 1995.

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Introduction

TRUSS-type large space structures (LSS) have been proposed with dimensions of many kilometers. Such structures (often called lattice structures) will be designed to be composed of identically constructed cell units, which are connected end-to-end to form a spatially periodic array. Structural and dynamic characteristics of the LSS must be predicted accurately during the initial design phase since they cannot be tested full scale in their operational environments prior to flight. Conventional finite element analysis of the LSS may require a significant amount of storage capacity and computing time to obtain reliable solutions because of its high structural flexibility and large size, especially in the dynamic analysis. Thus special techniques to cope with the large number of elements and nodes within such large truss-type structures are desired. Fortunately, most LSS have been expected to operate at relatively low frequencies. To perform that task most efficiently in this environment, many investigators have adopted the continuum modeling approach, which is known to provide very promising and practical solution method especially when the wavelength of a vibration mode spans many repeating cells of a truss-type structure.[1]

In the continuum modeling approach, a periodic truss-type structure is transformed into an equivalent continuum structural model by determining the equivalent structural properties of the continuum model from the appropriate relationship between the geometric and material properties of the truss-type structure and continuum models. Because there can be numerous concepts and ways to determine the equivalent structural properties in terms of the geometric and material properties of the original periodic truss-type structure, the continuum modeling approach will not be unique in nature, and there exist various continuum modeling methods developed in the past decades. To avoid an excessively long reference list, the reader is referred to the literature.[1–4]

The equivalence of the truss-type structure and the continuum model can be established by requiring that they have the same kinetic and strain energies,[1] constitutive relations,[2] static deformation characteristics,[3] or dispersion curves.[4] These continuum modeling methods may have their own shortcomings from the viewpoint of accuracy, versatility, or ease of use. Thus, recently a new continuum modeling, approach based on the concept of energy equivalence, has been developed, in which the finite element matrices for the structural elements within a representative truss cell are utilized for easy calculation of the kinetic and strain energies contained within the truss cell.[5–7]

The finite element based on continuum modeling method has proven quite easy to use and provides very reliable continuum models compared to other existing continuum modeling methods. Because of the approximation inherent in the conventional finite element formulation, however, the continuum model derived from the finite element matrices still will not be perfectly accurate. Thus, a more accurate and easy-to-use continuum modeling method is still desirable and in some sense mandatory. Hence, the objective of this study is to introduce a new continuum modeling method by requiring that a periodic truss-type beam and its equivalent continuum structural model take the same spectrally formulated transfer matrix.

Characteristics of Spectral Element

Doyle[8] introduced the spectral element to investigate wave propagation in structures. In contrast to the conventional finite element, the spectral element treats the mass distribution exactly and, therefore, dynamic behavior within each structural element is treated exactly. In this section, how a spectral formulation for a dynamic finite element of a beam can exactly model the distributed mass will be discussed.

The spectral formulation begins with the equations of motion of the beam including the inertia term as

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}=0 \tag{1}$$

The general solution of Eq. (1) can be given by the spectral representation of

$$v(x,t)=\sum_n \bar{v}_n(x,\omega_n)e^{i\omega_n t} \tag{2}$$

The spectral components $\bar{v}_n$ must satisfy

$$EI\frac{d^4\bar{v}_n}{dx^4}-\omega_n^2\rho A\bar{v}_n=0 \tag{3}$$

This has the simple solution

$$\begin{gathered}\bar{v}_n = Ae^{-ik_n x}+Be^{-k_n x}+Ce^{-ik_n(l-x)}+De^{-k_n(l-x)}\\ k_n=\sqrt{w_n}(\rho A/EI)^{\frac{1}{4}}\end{gathered} \tag{4}$$

where k_n is the wave number. Certainly $\bar{v}_n$ is dependent on the frequency ω_n. For a uniform beam element shown in Fig. 1, the dynamic stiffness matrix can be set up by first relating the coefficients A, B, C, and D to the nodal displacements in the form

$$\{A,B,C,D\}^T=[G]\{\bar{v}_1,\bar{\theta}_1,\bar{v}_2,\bar{\theta}_2\}^T \tag{5}$$

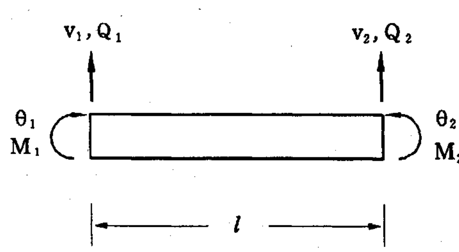

Fig. 1 Nodal forces and degrees of freedom for a uniform beam element.

After substitution of Eq. (5) into Eq. (4), the nodal forces are obtained by differention of Eq. (4) and then can be arranged to give

$$\begin{Bmatrix}\bar{Q}_1\\ \bar{M}_1\\ \bar{Q}_2\\ \bar{M}_2\end{Bmatrix}=\frac{EI}{l^3}\begin{bmatrix}\alpha_1 & \gamma_2 l & -\alpha_2 & \gamma_1 l\\ & \beta_1 l^2 & -\gamma_1 l & \beta_2 l^2\\ & & \alpha_1 & -\gamma_2 l\\ \text{symmetric} & & & \beta\end{bmatrix}\begin{Bmatrix}\bar{v}_1\\ \bar{\theta}_1\\ \bar{v}_2\\ \bar{\theta}_2\end{Bmatrix} \tag{6}$$

or, simply,

$$\{\bar{f}\}=[\bar{k}]\{\bar{u}\} \tag{7}$$

associated with the following definitions:

$$\begin{aligned}\alpha_1 &= (\cos k_b l\sinh k_b l+\sin k_b l\cosh k_b l)\\ &\quad\times(k_b l)^3/(1-\cos k_b l\cosh k_b l)\\ \alpha_2 &= (\sin k_b l+\sinh k_b l)(k_b l)^3/(1-\cos k_b l\cosh k_b l)\\ \beta_1 &= (-\cos k_b l\sinh k_b l+\sin k_b l\cosh k_b l)\\ &\quad\times(k_b l)/(1-\cos k_b l\cosh k_b l)\\ \beta_2 &= (-\sin k_b l+\sinh k_b l)(k_b l)/(1-\cos k_b l\cosh k_b l)\\ \gamma_1 &= (-\cos k_b l\cosh k_b l)(k_b l)^2/(1-\cos k_b l\cosh k_b l)\\ \gamma_2 &= (\sin k_b l\sinh k_b l)(k_b l)^2/(1-\cos k_b l\cosh k_b l)\\ k_b &= \sqrt{\omega}(\rho A/EI)^{\frac{1}{4}}\end{aligned} \tag{8}$$

The overbar superscript used in the preceding equations, which indicates the spectral components of the nodal displacements and forces, will be omitted for brevity in the following. The dynamic stiffness matrix $[k]$ can be numerically assembled in a completely analogous way to that used for the conventional finite element formulation. That is, the spectral elements can be assembled into the global system via the do loop over all frequency components as

$$F=KU \tag{9}$$

where K is the global dynamic stiffness matrix and F are the global nodal forces. The nodal forces are obtained by converting the time inputs to their spectrum through the use of fast Fourier transform (FFT). If the global nodal displacements U is solved for each frequency ω_n, then its dynamic behavior can be now reconstructed simply by using the inverse FFT. This is the overall procedure of the spectral formulation proposed by Doyle.[8]

Table 1 Comparison between the natural frequencies (rad/s) of a plane lattice structure by conventional FEM and SFEM

	Number of finite elements used				
	Conventional FEM				SFEM
Modes	40	80	120	160	40
1	322.70	321.20	321.11	321.10	321.09
2	775.66	754.21	753.06	752.85	752.74
3	963.37	919.75	917.20	916.75	916.51
4	991.95	932.23	929.37	928.86	928.59
5	1145.5	983.49	979.05	978.25	977.87
6	1247.9	1047.3	1042.0	1041.0	1040.5
7	1313.8	1140.4	1133.5	1132.2	1131.6
8	1397.1	1176.8	1171.0	1169.9	1169.4
9	1418.5	1235.5	1226.3	1224.6	1223.8
10	1609.7	1250.9	1242.5	1240.9	1240.1
15	1961.3	1394.4	1382.1	1379.6	1378.5
20	2353.2	1553.3	1537.3	1534.0	1532.4
25	2578.0	1875.2	1856.6	1851.7	1849.3
30	3824.9	2205.3	2188.9	2184.7	2182.4
35	4532.2	2548.3	2527.7	2522.4	2519.6

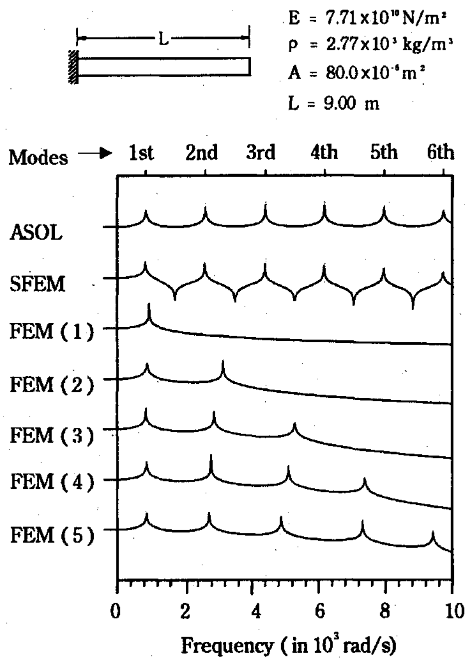

Fig. 2 Resonance spectra of an axial bar: analytical solution (ASOL), SFEM, and FEM using n finite elements [FEM(n)].

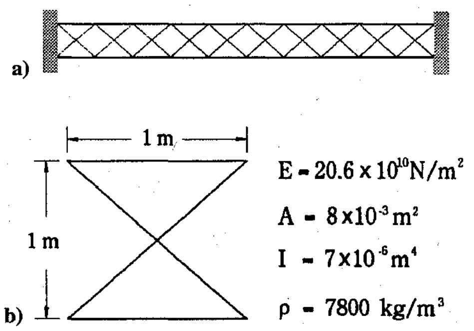

Fig. 3 Plane truss structure with 10 bays: a) original structure and b) representative truss cell.

To show the superior accuracy of the spectral finite element formulation in comparison with the conventional finite element formulation, two examples are investigated. Figure 2 first shows a comparison of the resonance spectra of an axial bar. The exact analytical resonances are compared with those by spectral finite element method (SFEM) and conventional finite element method (FEM). Only one finite element is used for SFEM, whereas n finite elements are used for FEM. It is obvious from Fig. 2 that SFEM yields exact resonances, whereas FEM approaches SFEM with an increasing number of finite elements used. Table 1 shows a comparison of the natural frequencies by SFEM and FEM for the plane lattice structure as shown in Fig. 3. The plane lattice structure consists of 40 beam elements. Table 1 also shows that the FEM converges to the SFEM as the number of finite elements used in the analysis increases.

Spectrally Formulated Transfer Matrix for a Truss Cell

In the preceding section, the spectral element is shown to be superior to the conventional finite element in respect to accuracy. In the spectral element method, as discussed in the preceding section, the dynamic stiffness matrix for a structural element (called the spectral element) is spectrally formulated to satisfy the boundary conditions specified by the nodal displacements and forces at its two end nodes. The spectrally formulated dynamic stiffness matrices for each structural element can be assembled as is done in the conventional finite element method. Since the spectral element treats the mass distribution exactly in contrast to the conventional finite element, the spectral element instead of the conventional finite element can be recommended for use in the continuum modeling process to improve the accuracy of the continuum structural model.

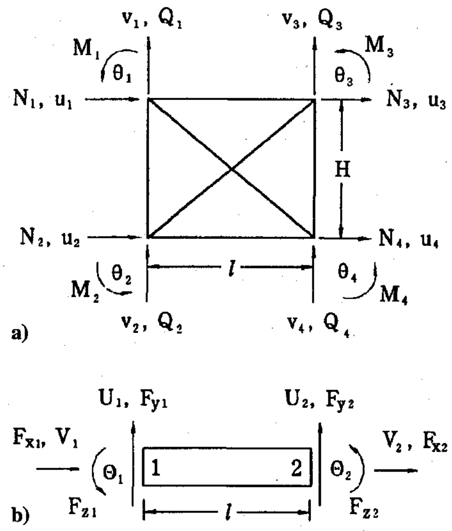

Fig. 4 Continuum representation of a) representative truss cell and b) its equivalent continuum ETB model.

As an example, a periodic truss-type beam will be considered in this study. A representative truss cell, which is isolated from the original truss-type beam, will be transformed into an equivalent continuum extended Timoshenko beam (ETB) element,[5] as shown in Fig. 4. The spectrally formulated dynamic stiffness matrices for the structural elements within a representative truss cell will depend on how the structural elements are assembled at the joints. If the structural elements within a truss cell are clamped at the joints, each structural element can be considered as the spectral beam element,[8] and thus the nodal forces vector $\boldsymbol{f}$ and the nodal displacements vector $\boldsymbol{d}$ for the truss cell can be defined as

$$
\begin{aligned}
\boldsymbol{d}_l &= \{u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2\}^T \\
\boldsymbol{d}_r &= \{u_3 \quad v_3 \quad \theta_3 \quad u_4 \quad v_4 \quad \theta_4\}^T \\
\boldsymbol{f}_l &= \{N_1 \quad Q_1 \quad M_1 \quad N_2 \quad Q_2 \quad M_2\}^T \\
\boldsymbol{f}_r &= \{N_3 \quad Q_3 \quad M_3 \quad N_4 \quad Q_4 \quad M_4\}^T
\end{aligned}
\tag{10}
$$

where u_i, v_i, and θ_i are the axial, transverse, and rotational nodal displacements at each node, respectively. N_i, Q_i, and M_i represent axial and transverse shear nodal forces, and the bending nodal moment, respectively. The subscripts l and r indicate the properties specified at the nodes on the left and right sections of the truss cell, as shown in Fig. 4. If the structural elements are pin joined at the joints, then each structural element can be considered as the spectral rod (or axial bar) element[8] and, accordingly, the rotational displacements and corresponding bending moments should be removed from Eqs. (10).

The spectrally formulated dynamic stiffness matrix for a spectral beam element of length l with two end nodes i and j can be derived as

$$
\begin{Bmatrix} \bar{N}_i \\ \bar{Q}_i \\ \bar{M}_i \\ \bar{N}_j \\ \bar{Q}_j \\ \bar{M}_j \end{Bmatrix}
= \frac{EI}{l^3}
\begin{bmatrix}
X & 0 & 0 & Y & 0 & 0 \\
 & \alpha_1 & \gamma_2 l & 0 & -\alpha_2 & \gamma_1 l \\
 & & \beta_1 l^2 & 0 & -\gamma_1 l & \beta_2 l^2 \\
 & & & X & 0 & 0 \\
 & \text{symmetric} & & & \alpha_1 & -\gamma_2 l \\
 & & & & & \beta_1 l^2
\end{bmatrix}
\begin{Bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{\theta}_i \\ \bar{u}_j \\ \bar{v}_j \\ \bar{\theta}_j \end{Bmatrix}
\tag{11}
$$

where

$$
\begin{aligned}
X &= \frac{l^3 A}{I}\frac{k_r}{\sin k_r l}\cos k_r l \\
Y &= -\frac{l^3 A}{I}\frac{k_r}{\sin k_r l} \\
k_r &= \omega(\rho A/EA)^{\frac{1}{2}}
\end{aligned}
\tag{12}
$$

In Eq. (11), α_i, β_i, γ_i, and k_b are defined as in Eqs. (8). For the spectral rod element, the dynamic stiffness matrix of Eq. (11) should be reduced to the 2×2 matrix retaining only the terms X and Y.

The spectrally formulated dynamic stiffness matrices derived for all structural elements within a truss cell can now be assembled in a completely analogous way to that used in the conventional finite element formulation to form a global dynamic stiffness matrix $Z(\omega)$ in the form

$$
\begin{Bmatrix} \boldsymbol{f}_l \\ \boldsymbol{f}_r \end{Bmatrix} = \begin{bmatrix} \boldsymbol{Z}_{ll}(\omega) & \boldsymbol{Z}_{lr}(\omega) \\ \boldsymbol{Z}_{rl}(\omega) & \boldsymbol{Z}_{rr}(\omega) \end{bmatrix} \begin{Bmatrix} \boldsymbol{d}_l \\ \boldsymbol{d}_r \end{Bmatrix} \tag{13}
$$

where $Z_{ij}(\omega)$ indicate the submatrices of $Z(\omega)$.

Now, assume that the equivalent continuum ETB element is located at the centerline of the representative truss cell and place the nodal forces and displacements, $\boldsymbol{F}$ and $\boldsymbol{D}$, at the two ends of the continuum ETB element, as shown in Fig. 4b. The nodal forces and displacements are named the continuum degrees of freedom in Ref. 5. The continuum degrees of freedom $\boldsymbol{F}$ and $\boldsymbol{D}$ are defined as

$$
\begin{aligned}
\boldsymbol{D}_l &= \{U_1 \quad V_1 \quad \Theta_1\}^T, & \boldsymbol{D}_r &= \{U_2 \quad V_2 \quad \Theta_2\}^T \\
\boldsymbol{F}_l &= \left\{F_{x_1} \quad F_{y_1} \quad F_{z_1}\right\}^T, & \boldsymbol{F}_l &= \left\{F_{x_2} \quad F_{y_2} \quad F_{z_2}\right\}^T
\end{aligned}
\tag{14}
$$

The nodal forces vector $\boldsymbol{f}$ and nodal displacements vector $\boldsymbol{d}$ for the truss cell can be linearly approximated in terms of the continuum degrees of freedom as

$$
\begin{aligned}
\begin{Bmatrix} \boldsymbol{d}_l \\ \boldsymbol{d}_r \end{Bmatrix} &= \begin{bmatrix} \boldsymbol{T}_d & \\ & \boldsymbol{T}_d \end{bmatrix} \begin{Bmatrix} \boldsymbol{D}_l \\ \boldsymbol{D}_r \end{Bmatrix} \\
\begin{Bmatrix} \boldsymbol{F}_l \\ \boldsymbol{F}_r \end{Bmatrix} &= \begin{bmatrix} \boldsymbol{T}_f & \\ & \boldsymbol{T}_f \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}_l \\ \boldsymbol{f}_r \end{Bmatrix}
\end{aligned}
\tag{15}
$$

where

$$
\boldsymbol{T}_f = \boldsymbol{T}_d^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -H/2 & 0 & 1 & H/2 & 0 & 1 \end{bmatrix} \tag{16}
$$

where H is shown in Fig 4a. For the case of a pin-joined truss cell, the matrix $\boldsymbol{T}_j$ (or $\boldsymbol{T}_d$) should be simplified by removing the third and sixth columns of it. Substituting Eqs. (15) into Eq. (13) yields

$$
\begin{aligned}
\begin{Bmatrix} \boldsymbol{F}_l \\ \boldsymbol{F}_r \end{Bmatrix} &= \begin{bmatrix} \boldsymbol{T}_f & \\ & \boldsymbol{T}_f \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}_{ll} & \boldsymbol{Z}_{lr} \\ \boldsymbol{Z}_{rl} & \boldsymbol{Z}_{rr} \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_d & \\ & \boldsymbol{T}_d \end{bmatrix} \begin{Bmatrix} \boldsymbol{D}_l \\ \boldsymbol{D}_r \end{Bmatrix} \\
&= \begin{bmatrix} \bar{\boldsymbol{Z}}_{ll} & \bar{\boldsymbol{Z}}_{lr} \\ \bar{\boldsymbol{Z}}_{rl} & \bar{\boldsymbol{Z}}_{rr} \end{bmatrix} \begin{Bmatrix} \boldsymbol{D}_l \\ \boldsymbol{D}_r \end{Bmatrix}
\end{aligned}
\tag{17}
$$

Equation (17) can be rewritten into the form of input-to-output relation as

$$
\begin{aligned}
\begin{Bmatrix} \boldsymbol{D}_r \\ \boldsymbol{F}_r \end{Bmatrix} &= \begin{bmatrix} -\bar{\boldsymbol{Z}}_{12}^{-1}\boldsymbol{Z}_{11} & -\bar{\boldsymbol{Z}}_{12}^{-1} \\ \bar{\boldsymbol{Z}}_{21} - \bar{\boldsymbol{Z}}_{22}\bar{\boldsymbol{Z}}_{12}^{-1}\bar{\boldsymbol{Z}}_{11} & -\bar{\boldsymbol{Z}}_{22}\bar{\boldsymbol{Z}}_{12}^{-1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{D}_l \\ \boldsymbol{F}_l \end{Bmatrix} \\
&= \boldsymbol{\Phi}_{\text{truss}}(\omega) \begin{Bmatrix} \boldsymbol{D}_l \\ \boldsymbol{F}_l \end{Bmatrix}
\end{aligned}
\tag{18}
$$

where $\boldsymbol{\Phi}_{\text{truss}}(\omega)$ is the transfer matrix for the representative truss cell. Note that the sign convention for $\boldsymbol{F}_l$ of Eq. (17) is that used universally in finite element analysis, and thus it should be changed to follow the sign convention for the mechanics of materials, as is done in Eq. (18).

Transfer Matrix for an ETB Element

A representative truss cell will be transformed into an equivalent continuum ETB element. The dynamic equations of motion for the continuum ETB model are as follows[5]:

$$
\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}x} &= \rho A\frac{\partial^2 u}{\partial t^2} + \rho R\frac{\partial^2 \theta}{\partial t^2}, \qquad \frac{\mathrm{d}Q}{\mathrm{d}x} = \rho A\frac{\partial^2 v}{\partial t^2} \\
\frac{\mathrm{d}M}{\mathrm{d}x} &= \rho I\frac{\partial^2 \theta}{\partial t^2} + \rho R\frac{\partial^2 u}{\partial t^2} - Q
\end{aligned}
\tag{19}
$$

with the force-displacement relations

$$
\begin{Bmatrix} N \\ Q \\ M \end{Bmatrix} = \begin{bmatrix} EA & C_{12} & C_{13} \\ C_{12} & GA & C_{23} \\ C_{13} & C_{23} & EI \end{bmatrix} \begin{Bmatrix} u' \\ v' - \theta \\ \theta' \end{Bmatrix} \tag{20}
$$

In Eqs. (19), ρ is the mass density per unit volume of the continuum ETB element and ρA, ρR, and ρI are the inertia properties defined as

$$
\rho A = \int_z \rho \,\mathrm{d}z, \qquad \rho R = \int_z \rho z \,\mathrm{d}z, \qquad \rho I = \int_z \rho z^2 \,\mathrm{d}z \tag{21}
$$

Assuming harmonic solutions, Eqs. (19) can be represented by the state equation as

$$
\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}x} = \boldsymbol{A}(\omega)\boldsymbol{y} \tag{22}
$$

where

$$
\boldsymbol{y} = \{u, v, \theta, N, Q, M\}^T \tag{23}
$$

and

$$
\boldsymbol{A}(\omega) = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & \begin{bmatrix} EA & C_{12} & C_{13} \\ C_{12} & GA & C_{23} \\ C_{13} & C_{23} & EI \end{bmatrix}^{-1} \\ \begin{matrix} -\rho A\omega^2 & 0 & -\rho R\omega^2 \\ 0 & -\rho A\omega^2 & 0 \\ -\rho R\omega^2 & 0 & -\rho I\omega^2 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{matrix} \end{bmatrix} \tag{24}
$$

Now, it is easy to derive the transition or transfer matrix $\boldsymbol{\Phi}(\omega)$ for the continuum ETB element from Eq. (22) as follows:

$$
\boldsymbol{\Phi}_{\text{ETB}}(\omega) = e^{\boldsymbol{A}(\omega)l} \tag{25}
$$

where l is the length of the continuum ETB element, which is equal to the length of a representative truss cell.

Derivation of Equivalent Structural Properties

The transfer matrices formulated for the representative truss cell and the continuum ETB element are required to be same for the derivation of the equivalent structural properties of the continuum ETB model. That is,

$$
\boldsymbol{\Phi}_{\text{truss}}(\omega) = \boldsymbol{\Phi}_{\text{ETB}}(\omega) \tag{26}
$$

Thus the matrix $A(\omega)$ of Eq. (24), which contains all equivalent structural properties of the continuum ETB model, can be readily represented in term of $\boldsymbol{\Phi}_{\text{truss}}$ from Eqs. (25) and (26) as

$$
\boldsymbol{A}(\omega) = (1/l)\ln \boldsymbol{\Phi}_{\text{truss}}(\omega) \tag{27}
$$

The transfer matrix Φ_{truss} for a representative truss cell should be numerically calculated from Eq. (18). Many numerical tests show that the right side of Eq. (27) certainly takes the same form as the matrix $A(\omega)$ of Eq. (24), that is,

$$\begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}^{-1} \\ \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{11} & 0 \\ b_{13} & 0 & b_{33} \end{bmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{matrix} \end{bmatrix} \tag{28}$$

In the matrix of Eq. (28), a_{ij} and b_{ij} are calculated by taking the natural logarithmic function of Φ_{truss} and then dividing the result by l. The equivalent structural properties are now determined by comparing the matrix components of $A(\omega)$ of Eq. (24) with the matrix components of Eq. (28) as follows:

$$EI = a_{11}, \qquad GA = a_{22}, \qquad EI = a_{33}$$

$$C_{12} = a_{12}, \qquad C_{13} = a_{13}, \qquad C_{23} = a_{23} \tag{29}$$

$$\rho A = -b_{11}/\omega^2, \qquad \rho I = -b_{33}/\omega^2, \qquad \rho R = -b_{13}/\omega^2$$

Note that the equivalent inertia properties for the present continuum ETB model may depend on the frequency, as can be seen from Eqs. (29).

Illustrative Example and Discussion

As an example, a periodic truss-type beam model shown in Fig. 5 will be considered for the continuum representation. The geometric and material properties of the truss-type beam are also given in Fig. 5. Every structural element within the truss-type beam will be considered to be clamped at the joints. Thus the spectral beam element is adopted herein to calculate the global dynamic stiffness matrix $Z(\omega)$ of Eq. (13).

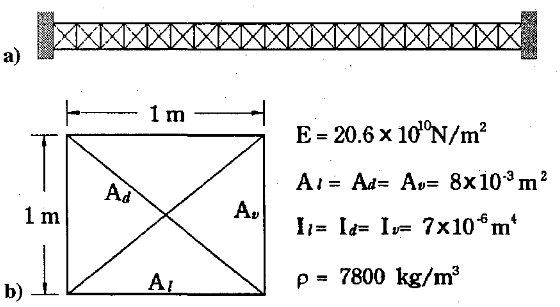

Fig. 5 Example of periodic truss beam with 20 bays: a) original structure and b) representative truss cell.

The equivalent structural properties of the continuum ETB model obtained from Eqs. (29) are illustrated in Fig. 6. The equivalent structural properties are found to be slightly dependent on frequency but become physically meaningless at the frequency near the first resonance ($\omega = 1700$ rad/s) of the structural elements within the truss cell. This shows that a continuum structural model will be valid only in the frequency region below the lowest resonant frequency of the truss-type structure.

In Table 2, the natural frequencies of the periodic truss-type beam with 20 repeating cells are compared for various analysis methods. The spectral element analysis[8] is proved to provide exact natural frequencies by showing that the natural frequencies obtained by the conventional finite element analysis gradually get closer to those derived by the spectral element analysis as the number of finite elements within a structural element increases. Comparing with the exact natural frequencies of the periodic truss-type beam obtained by the spectral element analysis, Table 2 certainly shows that the present continuum method (based on the spectral element formulation), in general, gives improved results in comparison with the previous continuum method of Ref. 5 (based on the conventional finite element formulation). The present continuum method is also shown to give very competitive results compared with the continuum method of Ref. 4 in which the dispersion curves are used. Especially for the fundamental bending vibration modes at which most LSS are expected to operate, the present continuum method is found to give sufficiently accurate natural frequencies.

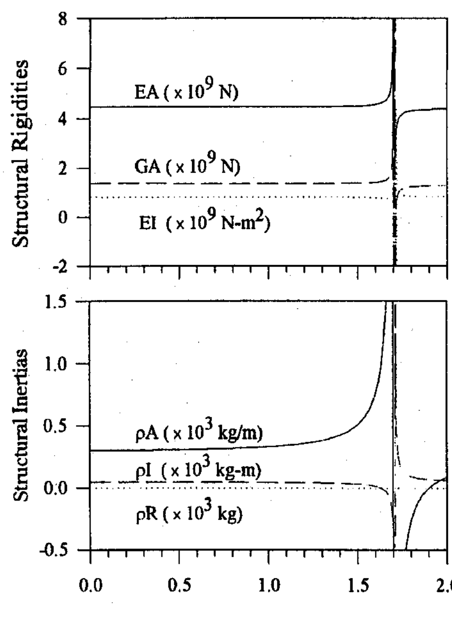

Fig. 6 Equivalent structural properties of the continuum ETB model.

Table 2 Comparison of the natural frequencies for the periodic truss-type beam

	Finite element analysis			Continuum modeling analysis		
Natural frequency, rad/s	One finite element	Two finite elements	Spectral finite element analysis (exact)	Spectral element based (present)	Finite element based (Ref. 5)	Dispersion curves based (Ref. 4)
$\omega_1(b)$[a]	81.245	81.225	81.223	81.249	80.581	81.060
$\omega_2(b)$	214.05	213.69	213.66	213.74	211.20	213.30
$\omega_3(b)$	397.58	395.29	395.13	395.48	390.26	395.90
$\omega_4(a)$	516.23	510.86	510.48	544.99	550.15	514.20
$\omega_5(b)$	618.69	610.01	609.37	610.74	604.57	615.90
$\omega_6(b)$	866.55	841.52	839.74	844.17	844.37	863.30
$\omega_7(a)$	989.00	931.36	927.95	1047.2	1100.3	991.80
$\omega_8(b)$	1131.1	1068.9	1064.7	1077.8	1102.3	1130.4
$\omega_9(b)$	1401.0	1183.6	1173.9	1287.7	1373.2	1411.7
...	...	...	...	...	...	...
$\omega_{16}(b)$	1656.0	1265.8	1254.0	1412.6	1653.3	1680.3

[a](a) and (b) indicate the axial and bending vibration modes, respectively.

Concluding Remarks

By use of the spectral element, a new continuum modeling method is introduced for the large truss-type periodic space structures that are expected to operate at relatively low-frequency vibration modes. Before adopting the spectral element in the continuum modeling procedure, the inherent accuracy of the spectral element is discussed in comparison with the conventional finite element, which has been used previously in a continuum modeling method. In this new continuum modeling method, the equivalent structural properties of a continuum model are derived by requiring that a representative cell of the original space structure and its equivalent continuum model have the same spectrally formulated transfer matrix. The spectrally formulated transfer matrix is reduced from the global dynamic stiffness matrix, which is obtained by assembling the element dynamic stiffness matrix of each spectral elements within a representative cell. The equivalent structural properties and vibration characteristics of a continuum ETB model by the present continuum method are compared with the results by other existing continuum methods. It is found that the present continuum method gives very reliable results compared to other continuum methods.

Acknowledgment

This work was supported by the Korea Science and Engineering Foundation under Grant 951-1009-037-2, which is gratefully appreciated.

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